

Abelian Dominance in Wilson Loops

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It has been conjectured that the Abelian projection of QCD is responsible for the confinement of color. Using a gauge independent definition of the Abelian projection (which does not employ any gauge fixing) we prove the Abelian dominance. In specific we prove that the gauge field configuration which contributes to the Wilson loop integral is precisely the gauge covariant part of the restricted potential, restricted by the Abelian projection. Our result strongly endorses the monopole condensation as the physical mechanism of the color confinement in QCD.

PACS numbers: 12.38.Aw, 12.38.Lg, 11.15.-q, 11.15.Ha

The confinement problem in QCD is perhaps one of the most difficult problems in theoretical physics. It has long been argued that the monopole condensation could provide the confinement of the color through a dual Meissner effect [1,2]. More explicitly it has been conjectured that the restricted part of QCD which comes from the “Abelian projection” of the theory to its maximal Abelian subgroup is responsible for the dynamics of the dual Meissner effect [2,3]. This conjecture, which asserts that in non-Abelian gauge theory only the degrees which correspond to its maximal Abelian subgroup should play the important role in the infra-red limit of the theory, is generally known as the “Abelian dominance”, and has been addressed by many authors in the literature [4,5]. But the actual proof of the Abelian dominance and the monopole condensation in the low energy limit of QCD has remained difficult.

A simple criterion for the confinement is given by the Wilson loop: if the vacuum expectation value of the Wilson loop satisfies the area law for a large current loop of a colored object, the confinement could be assured. This suggests that the Abelian dominance could be tested through the Wilson loop calculation. In this direction, a remarkable progress has been made by the numerical simulation during the last decade. In fact, the lattice calculation has confirmed the Abelian dominance and showed that the dominant contribution to the string tension in QCD (actually 92%) comes from the Abelian projection of the theory [6,7]. If this is so, one should be able to prove the Abelian dominance theoretically, independent of the numerical simulation. The purpose of this Letter is to provide a theoretical proof of the Abelian dominance in the Wilson loop calculation. In specific we prove that *it is the Abelian projection of the full connection, more precisely the gauge covariant part of the Abelian projection, which contributes to the Wilson loop integral*. Furthermore we show that *the Wilson loop integral can be expressed as the generating functional*

of the restricted QCD which describes the dual dynamics of the non-Abelian gauge theory. This strongly endorses the magnetic condensation and dual Meissner effect as the dynamical mechanism for the confinement of color in QCD.

Consider a non-Abelian gauge theory of a given gauge group G ,

$$\mathcal{L} = -\frac{1}{4}\vec{F}_{\mu\nu}^2, \quad (1)$$

where $\vec{F}_{\mu\nu}$ is the field strength. To prove the Abelian dominance in the infra-red limit, one must first know how to project out and separate the connection which corresponds to the maximal Abelian subgroup H from the full non-Abelian connection of the group G . For $SU(2)$ this means that one should project out the $U(1)$ component of the connection in a gauge-independent way. To do this, one must select the $U(1)$ direction at each space-time point, and make a gauge-independent projection of the connection which contains only the $U(1)$ degree. This can be done by introducing a unit iso-triplet scalar field $\hat{n}(x)$ which transforms covariantly under the gauge transformation, and insisting that \hat{n} remains unchanged under the parallel transport [2]. So we require \hat{n} to be a covariant constant,

$$D_\mu \hat{n} = 0 \quad (\hat{n}^2 = 1). \quad (2)$$

Clearly \hat{n} selects the $U(1)$ direction at each space-time point, and the parallel transport (2) provides the desired Abelian projection of the full connection \vec{A}_μ ,

$$\vec{A}_\mu \longrightarrow \hat{A}_\mu = A_\mu \hat{n} - \frac{1}{g} \hat{n} \times \partial_\mu \hat{n}, \quad (3)$$

where $A_\mu = \hat{n} \cdot \vec{A}_\mu$ is the “naive” Abelian component (the electric potential) of the connection. This \hat{A}_μ is the restricted connection we introduced some time ago

[2,3]. It has many interesting features. First, \hat{n} being gauge covariant, the projection (3) is obviously gauge-independent. Moreover, \hat{A}_μ retains the full $SU(2)$ gauge degrees of freedom even though it is clearly restricted. This is because the gauge-independent projection still makes it an $SU(2)$ connection. Indeed, under an arbitrary gauge transformation specified by an infinitesimal parameter $\vec{\theta}$, one has

$$\delta \hat{n} = -\vec{\theta} \times \hat{n}, \quad \delta \vec{A}_\mu = \frac{1}{g} D_\mu \vec{\theta}, \quad (4)$$

which guarantees

$$\delta \hat{A}_\mu = \frac{1}{g} \hat{D}_\mu \vec{\theta} = \frac{1}{g} (\partial_\mu \vec{\theta} + g \hat{A}_\mu \times \vec{\theta}). \quad (5)$$

More importantly, \hat{A}_μ retains the full topological characteristics of the original non-Abelian potential. In fact, the isolated singularities of \hat{n} defines $\pi_2(S^2)$ which describes the non-Abelian monopoles [2,8]. Indeed \hat{A}_μ with $A_\mu = 0$ and $\hat{n} = \hat{r}$ describes precisely the Wu-Yang monopole [9]. Besides, with the S^3 compactification of R^3 , \hat{n} defines the Hopf invariant $\pi_3(S^2)$ which describes the topologically distinct vacua [10].

The above discussion implies that there exists a subclass of the non-Abelian gauge theory, the restricted gauge theory, which contains only the Abelian projection which nevertheless has the full non-Abelian gauge degrees of freedom [2,3]. To understand this, notice that with the Abelian projection (3) one has

$$\begin{aligned} \hat{F}_{\mu\nu} &= (F_{\mu\nu} + H_{\mu\nu})\hat{n}, \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, \\ H_{\mu\nu} &= -\frac{1}{g} \hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}) = \partial_\mu C_\nu - \partial_\nu C_\mu, \end{aligned} \quad (6)$$

where the last equality follows from $\partial_\mu \tilde{H}_{\mu\nu} = 0$ (except for the isolated singularities of \hat{n}). To find the potential C_μ let

$$S = \exp(-t_3 \gamma) \exp(-t_2 \alpha) \exp(-t_3 \beta),$$

where t_i are the adjoint representation of the $SU(2)$ generators, and let

$$\hat{n}_i = S^{-1} \hat{e}_i \quad (i = 1, 2, 3), \quad (7)$$

where $\hat{e}_1 = (1, 0, 0)$, $\hat{e}_2 = (0, 1, 0)$, and $\hat{e}_3 = (0, 0, 1)$. Now we identify \hat{n} to be \hat{n}_3 ,

$$\hat{n} = \hat{n}_3 = (\sin \alpha \cos \beta, \sin \alpha \sin \beta, \cos \alpha). \quad (8)$$

Then under the gauge transformation S , one has

$$\begin{aligned} \hat{A}_\mu &\longrightarrow (A_\mu + C_\mu) \hat{e}_3, \\ \hat{F}_{\mu\nu} &\longrightarrow (F_{\mu\nu} + H_{\mu\nu}) \hat{e}_3, \end{aligned} \quad (9)$$

where

$$\begin{aligned} C_\mu &= \frac{1}{g} (\cos \alpha \partial_\mu \beta + \partial_\mu \gamma) \\ &= -\frac{1}{g} \hat{n}_1 \cdot \partial_\mu \hat{n}_2. \end{aligned} \quad (10)$$

This shows that C_μ describes precisely the Dirac's monopole potential around the isolated singularities of \hat{n} . So the restricted gauge theory describes the dual dynamics of the color charge and the non-Abelian monopole.

With the Abelian projection it is easy to obtain the full connection \vec{A}_μ . Since the connection space forms an affine space, one can obtain an arbitrary non-Abelian connection \vec{A}_μ simply by adding a gauge-covariant vector field \vec{X}_μ to \vec{A}_μ ,

$$\vec{A}_\mu = A_\mu \hat{n} - \frac{1}{g} \hat{n} \times \partial_\mu \hat{n} + \vec{X}_\mu = \hat{A}_\mu + \vec{X}_\mu. \quad (11)$$

Notice that \vec{X}_μ must be orthogonal to \hat{n} . From (4) and (5) one can easily confirm that \vec{X}_μ indeed forms a covariant multiplet,

$$\delta \vec{X}_\mu = -\vec{\theta} \times \vec{X}_\mu. \quad (12)$$

With (11) one has

$$\vec{F}_{\mu\nu} = \hat{D}_\mu \vec{X}_\nu - \hat{D}_\nu \vec{X}_\mu + (F_{\mu\nu} + H_{\mu\nu} + X_{\mu\nu}) \hat{n}, \quad (13)$$

where $X_{\mu\nu} = g \hat{n} \cdot (\vec{X}_\mu \times \vec{X}_\nu)$. Furthermore with

$$\vec{X}_\mu = X_\mu^1 \hat{n}_1 + X_\mu^2 \hat{n}_2,$$

one has

$$\begin{aligned} \hat{D}_\mu \vec{X}_\nu &= [\partial_\mu X_\nu^1 - g(A_\mu + C_\mu) X_\nu^2] \hat{n}_1 \\ &\quad + [\partial_\mu X_\nu^2 + g(A_\mu + C_\mu) X_\nu^1] \hat{n}_2. \end{aligned} \quad (14)$$

So one could express the Lagrangian explicitly in terms of A_μ , \hat{n} , and \vec{X}_μ ,

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} (F_{\mu\nu} + H_{\mu\nu} + X_{\mu\nu})^2 \\ &\quad - \frac{1}{4} |\hat{D}_\mu \vec{X}_\nu - \hat{D}_\nu \vec{X}_\mu|^2. \end{aligned} \quad (15)$$

Clearly this describes a $U(1)$ gauge theory coupled to a charged vector field \vec{X}_μ . But the important point here is that the $U(1)$ potential is given by $(A_\mu + C_\mu)$, not A_μ . The corresponding equations of motion is given by

$$\begin{aligned} \partial_\mu (F_{\mu\nu} + H_{\mu\nu} + X_{\mu\nu}) &= -g \hat{n} \cdot \vec{X}_\mu \times (\hat{D}_\mu \vec{X}_\nu - \hat{D}_\nu \vec{X}_\mu), \\ \hat{D}_\mu (\hat{D}_\mu \vec{X}_\nu - \hat{D}_\nu \vec{X}_\mu) &= g (F_{\mu\nu} + H_{\mu\nu} \\ &\quad + X_{\mu\nu}) \hat{n} \times \hat{X}_\mu. \end{aligned} \quad (16)$$

This allows us to interpret \vec{X}_μ as “the valence gluon” which plays the role of a colored source of the restricted theory [2,3]. More importantly this implies that \vec{X}_μ , just like the quarks, could play only a minor role (if at all) in

the confinement mechanism. It represents simply another colored source which has to be confined itself. This is the reason why only the restricted connection (3) should play the dominant role in the Wilson loop calculation.

Notice that under the inverse gauge transformation S^{-1} which rotates \hat{e}_3 to \hat{n} , one must have

$$\begin{aligned}\hat{A}_\mu &= (A_\mu + C_\mu)\hat{n} + \frac{1}{g}\text{tr}(-\frac{1}{2}\vec{t}S^{-1}\partial_\mu S) \\ &= A_\mu\hat{n} - \frac{1}{g}\hat{n} \times \partial_\mu\hat{n}.\end{aligned}\quad (17)$$

More importantly under the gauge transformation (3) one has

$$\begin{aligned}\delta(A_\mu + C_\mu) &= \delta(\hat{n} \cdot \vec{A}_\mu) - \frac{1}{g}\delta(\hat{n}_1 \cdot \partial_\mu\hat{n}_2) \\ &= 0.\end{aligned}\quad (18)$$

This shows that $(A_\mu + C_\mu)\hat{n}$ is the gauge covariant part of the restricted connection. This in turn allows us to identify the gauge covariant part $\vec{A}_\mu^{(c)}$ of the full connection,

$$\begin{aligned}\vec{A}_\mu &= (A_\mu + C_\mu)\hat{n} + \vec{X}_\mu + \frac{1}{g}\text{tr}(-\frac{1}{2}\vec{t}S^{-1}\partial_\mu S) \\ &= \vec{A}_\mu^{(c)} + \frac{1}{g}\text{tr}(-\frac{1}{2}\vec{t}S^{-1}\partial_\mu S).\end{aligned}\quad (19)$$

This observation will become important in the Wilson loop calculation.

Now we are ready to discuss Wilson loop along a closed curve C . The Wilson loop integral, although conceptionally simple, has remained very difficult to carry out. But an important step to simplify the integral was made by Diakonov and Petrov, who showed that integral can be expressed as a functional integral over *all* gauge transformations $S(t)$ along the loop [11],

$$\begin{aligned}W(C) &= \text{tr}P \exp[-\oint \mathcal{A}_\mu dx^\mu] \\ &= \int \mathcal{D}S(t) \exp[\frac{1}{2}\oint \text{tr}t_3(S\mathcal{A}_\mu S^{-1} \\ &\quad + \frac{1}{g}S\partial_\mu S^{-1})dx^\mu],\end{aligned}\quad (20)$$

where $\mathcal{A}_\mu = \vec{A}_\mu \cdot \vec{t}$. Now from (19) we have

$$\begin{aligned}-\frac{1}{2}\text{tr} [t_3(S\mathcal{A}_\mu S^{-1} + \frac{1}{g}S\partial_\mu S^{-1})] \\ &= -\frac{1}{2}\text{tr}[S^{-1}t_3S(\mathcal{A}_\mu - \frac{1}{g}S^{-1}\partial_\mu S)] \\ &= \hat{n} \cdot \vec{A}_\mu^{(c)} \\ &= A_\mu + C_\mu.\end{aligned}\quad (21)$$

This makes us to identify the gauge field configuration which is relevant to the Wilson loop integral. *It is indeed*

the restricted gauge potential, more precisely the gauge invariant part of the restricted potential, that contributes to the Wilson loop integral. This proves the Abelian dominance, which confirms the conjecture that the dynamics of the restricted gauge theory (or the dual gauge theory) is responsible for the color confinement in QCD.

With (21) we can now obtain the desired expression for the Wilson loop,

$$\begin{aligned}\langle W(C) \rangle &= \int \mathcal{D}A_\mu \mathcal{D}\hat{n} \mathcal{D}\vec{X}_\mu \exp[-\frac{1}{4}\int \vec{F}_{\mu\nu}^2 d^4x \\ &\quad + \oint (A_\mu + C_\mu)dx^\mu].\end{aligned}\quad (22)$$

By integrating out the \vec{X}_μ degrees of freedom, one can express the integral as the vacuum average of (20) over the effective Lagrangian $\hat{\mathcal{L}}_{eff}$ of the restricted QCD,

$$\begin{aligned}\langle W(C) \rangle &= \int \mathcal{D}A_\mu \mathcal{D}\hat{n}_\mu \exp[-\int \hat{\mathcal{L}}_{eff} d^4x \\ &\quad + \oint (A_\mu + C_\mu)dx^\mu] \\ &= \int \mathcal{D}A_\mu \mathcal{D}\hat{n}_\mu \exp\{-\int [\hat{\mathcal{L}}_{eff} \\ &\quad + (A_\mu + C_\mu)j^\mu]d^4x\},\end{aligned}\quad (23)$$

where

$$j^\mu = \int \delta^4(x - z(t)) \frac{dz^\mu}{dt} dt.$$

On the other hand, since the integral (20) does not contain \vec{X}_μ , one might as well use the Lagrangian for the restricted theory in the integral (22), putting $\vec{X}_\mu = 0$ and skipping the \vec{X}_μ integral. So one could write

$$\begin{aligned}\langle W(C) \rangle &= \int \mathcal{D}A_\mu \mathcal{D}\hat{n}_\mu \exp\{-\int [\frac{1}{4}\hat{F}_{\mu\nu}^2 \\ &\quad + (A_\mu + C_\mu)j^\mu]d^4x\} \\ &= \int \mathcal{D}A_\mu \mathcal{D}C_\mu \exp\{-\int [\frac{1}{4}(F_{\mu\nu} + H_{\mu\nu})^2 \\ &\quad + (A_\mu + C_\mu)j^\mu]d^4x\},\end{aligned}\quad (24)$$

where we have changed the variable \hat{n} by C_μ (it is understood that $\mathcal{D}C_\mu$ includes the Jacobian for the change of variable). This shows that *one can reduce the evaluation of the Wilson loop integral to the evaluation of the generating functional of the restricted gauge theory for the gauge invariant external current j_μ .* This is really remarkable, but perhaps not so surprising. It has been known that the evaluation of the Wilson loop integral could be related to the evaluation of a gauge invariant part of the generating functional which is invariant under the gauge transformation of the external current \vec{j}_μ [11]. Our result not only confirms this but more importantly drastically simplifies the integral, which is made possible

by identifying the field configuration which contributes to the Wilson loop.

The physics behind our main result (24) is unmistakable. Clearly the electric potential A_μ generates the usual Coulomb potential. But the magnetic potential C_μ , with the magnetic condensation of the vacuum $\langle H_{\mu\nu}^2 \rangle \neq 0$, should generate the linear potential which confines the color. The important point here is the mechanism of the dynamical symmetry breaking and the generation of the confinement scale. In our formulation the symmetry breaking is guaranteed by (18). Notice that A_μ and C_μ separately enjoys the gauge symmetry of the maximal Abelian subgroup H , but $A_\mu + C_\mu$ together no longer has the symmetry. With the disappearance of the $U(1)$ symmetry $A_\mu + C_\mu$ must acquire a mass through the quantum correction, which in turn triggers the dual Meissner effect.

We close with the following remarks:

1) It must be emphasized that our definition of the Abelian projection is different from the others, although the underlying physics behind it is probably the same. In the popular definition of 't Hooft, the Abelian projection is regarded as a partial gauge fixing (called the maximal Abelian gauge) of G/H degrees [4,5]. In another definition which does not employ any gauge fixing, the projection is not supposed to depend on any particular set of field configurations [12]. In comparison our Abelian projection (3) selects a particular set of field configurations (the restricted potential), and is explicitly gauge independent. Furthermore after the projection the restricted potential enjoys the full gauge degrees of freedom.

2) We have shown that the valence part of the potential \vec{X}_μ does not contribute to the Wilson loop. Physically this is because it represents a colored source which has to be confined itself. There is another intuitive reason why it can be neglected in the infra-red limit. As a gauge covariant multiplet it must acquire a mass through the quantum correction. So in the low energy limit it could be treated as a heavy non-propagating source, and safely be neglected.

3) It must be clear that it is the topological degree of \hat{n} (described by C_μ) which is responsible for the confinement. In the Wilson loop integral this topological degree is naturally (and rightfully) included in the functional integral, but it must be emphasized that at the classical level \hat{n} can *not* be treated as a dynamical (i.e., propagating) field. This is because one can always remove \hat{n} with the gauge transformation S , at least locally. In this connection Faddeev and Niemi has recently made an interesting ansatz, $\vec{X}_\mu = f_1 \partial_\mu \hat{n} + f_2 \hat{n} \times \partial_\mu \hat{n}$, and conjectured that \hat{n} should be counted as a dynamical degree in the infra-red limit [13]. But notice that this ansatz, although clearly consistent, does not describe the most general connection. In fact this ansatz does not make \hat{n} dynamical at the classical level. It becomes dynamical in the effective

Lagrangian only after the \vec{X}_μ integration.

A more detailed discussion, including the derivation of the effective Lagrangian for the restricted gauge theory and the generalization to an arbitrary group G , will be given in a forthcoming paper [14].

The author thanks Professor C. N. Yang for fruitful discussions and encouragements, and appreciates interesting discussions with L. Faddeev, K. Fujikawa, A. Niemi, and H. Toki. The work is supported in part by Korean Science and Engineering Foundation through Center for Theoretical Physics, and by Korea Research Foundation through Project 1998-015-D0054.

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